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SYSTEMATIC ERRORS OF A HORIZON SENSOR DUE TO THE NONSPHERICITY OF THE PLANET

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SYSTEMATIC ERRORS OF A HORIZON SENSOR DUE TO THE NONSPHERICITY OF THE PLANET

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ABSTRACT. Detailed analysis of the systematic errors of devices for determining the orientation and position of a spacecraft near a planet from observations of the planet's disk. The deterministic and probabilistic approaches to the problem of determining the systematic errors are compared. The results are extended to include a vehicle moving along an arbitrarily inclined orbit.

Statement of Problem

The systematic errors of a horizon scanning system are caused by the geometric features of a planet and the physical properties of its atmosphere. /285*
Obviously, when there are clouds or mountains, the position of the edge of the planetary disk (at least in the range of the visible spectrum) will not be determined precisely. Atmospheric refraction and scattering can also cause errors. Some of the above-stated errors are analyzed in [2, 3]. One of the systematic errors is related to the nonsphericity of the planet. The following expressions were obtained in [1] for the above-stated errors of the device in the determination of the direction to the center of the planet (in the plane of the meridian) and distance to it:

$$\Delta\gamma = \frac{e^2a^2}{2\rho^2}\sin\varphi\cos\varphi, \tag{1}$$

$$\Delta\rho = \frac{e^2a^2}{2\rho}\left[1 + \frac{(\rho^2 - 3a^2)}{2a^2}\cos^2\varphi\right], \tag{2}$$

where a is the equatorial radius of the planet, e is the eccentricity of the spheroid, ρ is the modulus of the vector radius of the object, ϕ is the planetocentric latitude of the position of the object.

The coefficient 1/2 in expression (1) is erroneous. This error[1] is the consequence of the error found in [1] in the determination of certain intermediate values $\delta\overline{\Theta}_{12}$ and $\delta\overline{\Theta}_{34}$.

The corrected expression for the error of the instrument in the determination of the direction to the center of the planet (in the plane of the meridian) will be

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[1]The possibility of such an error was discussed earlier in [3].
*Numbers in the margin indicate pagination in the foreign text.

$$\Delta\gamma = \frac{e^2a^2}{\rho^2}\sin\varphi\cos\varphi. \qquad (3)$$

An assumption was made in [2] that angle ϕ is a random value, distributed according to the law of uniform density (this corresponds to the case of motion of the object in a polar circular orbit). The function of the distribution $G(y)$ of the random value $\Delta\gamma$, where y is a flowing value of error $\Delta\gamma$, was obtained /286

$$G(y) = \frac{1}{\pi}\arccos\frac{(-y)}{K}.$$

In this case the expression for the limit error of the instrument in the determination of the direction to the center of the planet with a probability of 95%, with consideration of (3), will be[1]

$$\Delta\gamma(95\%) = 0{,}494\frac{e^2a^2}{\rho^2}. \qquad (4)$$

According to [2], the function of distribution $F(z)$ of the random value $\Delta\rho$ has the form

$$F(z) = \frac{2}{\pi}\arcsin\sqrt{\frac{\Delta\rho - A}{B}}, \qquad (5)$$

where

$$A = \frac{e^2a^2}{2\rho}, \quad B = \frac{e^2}{4\rho}(\rho^2 - 3a^2), \qquad (6)$$

and the limit value of error in the determination of the radial distance with a probability of 90% is determined by the expression

$$\Delta\rho(90\%) = 0{,}243\frac{e^2a^2}{\rho^2}, \qquad (7)$$

i.e. the error decreases as the altitude of the flight increases.

It will be shown below that expression (5) is valid only up to the value $\rho > a\sqrt{3}$, and relation (7) is erroneous. Moreover, we will examine the case of arbitrary orbital inclination to the equatorial plane.

[1] In [2], $K = e^2a^2/4\rho^2$, which is incorrect (see footnote on p. 1). Formula (4) was found after the corresponding correction.

From expression (2), with consideration of (6), we have

$$\Delta\rho = A + B\cos^2\varphi. \tag{8}$$

We will assume that the values of angle ϕ are distributed uniformly in the range of $+\pi/2$ to $-\pi/2$, which is valid for an apparatus moving in a polar circular orbit.

We have

$$f(x) = \begin{cases} \dfrac{1}{\pi} & \text{for } |x| < \dfrac{\pi}{2}, \\ 0 & \text{for } |x| > \dfrac{\pi}{2}, \end{cases} \tag{9}$$

where f(x) is the probability density of the random value ϕ.

We find now the density of probability g(y) of the random value $Y = \cos^2\phi$.

We have

$$g(y) = \begin{cases} \dfrac{1}{\pi\sqrt{y}\sqrt{1-y}} & \text{for } 0 < y < 1, \\ 0 & \text{for } y < 0 \text{ and } y > 1. \end{cases} \tag{10}$$

We will find the distribution function F(z) of the random value $\Delta\rho$, where, with consideration of (8) /287

$$\Delta\rho = A + BY. \tag{11}$$

We will now examine two cases separately:

Case 1 ($B > 0$, which corresponds to $\rho > a\sqrt{3}$). With consideration of expressions (10) and (11), the density of probability g(z) of the random value $\Delta\rho$ is determined by the relation

$$g(z) = \frac{1}{\pi B\sqrt{\dfrac{z-A}{B}}\sqrt{1-\dfrac{z-A}{B}}}.$$

Then the distribution function F(z) of the random value $\Delta\rho$ will be

$$F(z)=\frac{1}{\pi B}\int_{A}\frac{dz}{\sqrt{\frac{z-A}{B}}\sqrt{1-\frac{z-A}{B}}},$$

where the lower limit of the integral is defined such that when $B > 0$, according to (11), $z_{min} = A$, from which, finally, we have

$$F(z)=\frac{2}{\pi}\arcsin\sqrt{\frac{z-A}{B}}. \qquad (12)$$

This result coincides with the function obtained in [2].

Case 2 ($B < 0$, which corresponds to $\rho < a\sqrt{3}$). The probability density g(z) in this case will be

$$g(z)=\frac{1}{\pi|B|\sqrt{\frac{z-A}{B}}\sqrt{1-\frac{z-A}{B}}},$$

hence

$$F(z)=\frac{1}{\pi|B|}\int_{A+B}^{z}\frac{dz}{\sqrt{\frac{z-A}{B}}\sqrt{1-\frac{z-A}{B}}},$$

where the lower limit of the integral is defined such that when $B < 0$, according to (11), $z_{min} = A + B$.

Finally, we have

$$F(z)=1-\frac{2}{\pi}\arcsin\sqrt{\frac{z-A}{B}}. \qquad (13)$$

This particularly important, in the practical sense, case (for the Earth, for instance, at altitudes up to 1,047 km) is not discussed in [2].

From expressions (12) and (13), with consideration of (11), we obtain with probability 95% the general expressions for the limiting errors of the instrument in the determination of the radial distance to the center of the planet:

$$\Delta\rho = A + 0{,}0062\ B \quad \text{for} \quad a \leqslant \rho \leqslant a\sqrt{3}, \tag{14}$$

$$\Delta\rho = A + 0{,}9938\ B \quad \text{for} \quad \rho \geqslant a\sqrt{3}, \tag{15}$$

where for $\rho = a\sqrt{3}$ from (6), $\Delta\rho = A$.

It must be stressed here that in [2] for this case, the following expression was found: $\Delta\rho(90\%) = 0.243(e^2a^2/\rho)$, which leads to an erroneous conclusion concerning the reduction of error for the case $\rho > a\sqrt{3}$ as the flight altitude increases (cf. (14) and 15)).

Errors of Horizon Sensor in the Case of Inclinced Orbit /288

Examination of the case of motion of an object in an inclined circular orbit is important.

We have the following relation

$$\sin\varphi_1 = \sin\varphi \sin\alpha,$$

where ϕ is the latitude of the position of the object, which moves in a polar circular orbit at an angle of 90° - α to the inclined orbit; ϕ_1 is the latitude of the position of an object moving at the same angular velocity in the inclined circular orbit; α is the angle of inclination of the orbit to the equatorial plane.

We find the density of probability $g(y)$ of the random value Y, where $Y = \sin^2\phi$,

$$g(y) = \begin{cases} \dfrac{1}{\pi\sqrt{y}\sqrt{1-y}} & \text{for } 0 < y < 1, \\ 0 & \text{for } y < 0,\ y > 1. \end{cases} \tag{16}$$

Since $\sin^2\phi_1 = \sin^2\phi \sin^2\alpha$, then, by denoting $Z = \sin^2\phi_1$, with consideration of (16), we have

$$g(z) = \begin{cases} \dfrac{1}{\pi\sqrt{z}\sqrt{\sin^2\alpha - z}} & \text{for } \sin^2\alpha < z < 1, \\ 0 & \text{for } z < \sin^2\alpha \text{ and } z > 1, \end{cases} \tag{17}$$

where z is the flowing value of the random value Z.

With consideration of (17), the probability density g(u) of the random value U, where $U = \cos^2 \phi_1$, will be

$$g(u)=\begin{cases}\dfrac{1}{\pi\sqrt{1-u}\sqrt{u-\cos^2\alpha}} & \text{for } \cos^2\alpha<u<1,\\ 0 & \text{for } u<\cos^2\alpha,\ u>1,\end{cases} \tag{18}$$

where u is the flowing value of the random value U.

For the inclined orbit, we have, from (8),

$$\Delta\rho = A + B\cos^2\varphi_1, \tag{19}$$

hence, with consideration of (18), the probability density f(z) of the random value $\Delta\rho$ will be

$$f(z)=\frac{dz}{\pi|B|\sqrt{1-\dfrac{y-A}{B}}\sqrt{\dfrac{y-A}{B}-\cos^2\alpha}}, \tag{20}$$

where z is the flowing value of the random value $\Delta\rho$.

We will find the function of distribution F(z).

We will consider two cases.

Case 1 ($B > 0$, which corresponds to $\rho > a\sqrt{3}$).

From expression (20) we have

$$F(z)=\frac{1}{\pi B}\int\limits_{A+B\cos^2\alpha}^{z}\frac{dz}{\sqrt{1-\dfrac{y-A}{B}}\sqrt{\dfrac{y-A}{B}-\cos^2\alpha}},$$

where the lower boundary of the integral is defined such that when $B > 0$, from expression (19), $z_{min} = A + B\cos^2\alpha$, hence /289

$$F(z)=\frac{2}{\pi}\arcsin\left(\frac{\sqrt{\dfrac{z-A}{B}-\cos^2\alpha}}{\sin\alpha}\right). \tag{21}$$

Case 2 ($B < 0$, which corresponds to $a < \rho < a\sqrt{3}$).

We have

$$F(z)=\frac{1}{\pi|B|}\int_{A+B}^{z}\frac{dz}{\sqrt{1-\frac{z-A}{B}}\sqrt{\frac{z-A}{B}-\cos^2\alpha}}=$$

where the lower boundary of the integral is defined such that when B < 0, from relation (19), z_{min} = A + B, hence

$$F(z)=1-\frac{2}{\pi}\arcsin\left(\frac{\sqrt{\frac{z-A}{B}-\cos^2\alpha}}{\sin\alpha}\right). \tag{22}$$

The final expressions for the limit values of the errors of the instrument in the determination of the radial distance to the center of the planet with a probability of 95% from (21) and (22), with consideration of (19), will be

$$\Delta\rho(95\%)=A+B(1-0{,}9938\sin^2\alpha)\quad \text{for}\quad a\leqslant\rho\leqslant a\sqrt{3}, \tag{23}$$

$$\Delta\rho(95\%)=A+B(\cos^2\alpha+0{,}9938\sin^2\alpha)\quad \text{for}\quad \rho\geqslant a\sqrt{3}. \tag{24}$$

In particular, when α = 90°,

$$\Delta\rho(95\%)=A+0{,}0062\,B\quad \text{for}\quad a\leqslant\rho\leqslant a\sqrt{3},$$

$$\Delta\rho(95\%)=A+0{,}9938\,B\ \text{for}\ \rho\geqslant a\sqrt{3},$$

which corresponds to the relations found earlier, namely (14) and (15).

Limit Errors of Horizon Sensor

Comparison of the original expressions (2) and (3) with the functions (4), (23), and (24), indicates that the relations obtained with a probability of 95% for the limit errors of the horizon center in the determination of the direction to the center of the planet and distance to it, due to the effect of the nonsphericity of the planet, is quite close in value to the limit errors. This can be expected, considering that the random value ϕ (latitude of the position of the object) is confined to +π/2 to -π/2 and was subordinate to the law of uniform density. Then, with consideration of (3), expressions for the limit errors in the determination of the direction to the center of the planet will be

a) for an orbit inclined to the equator at an angle 0°< α < 45°,

$$\Delta\gamma_{1\,lm}=\frac{e^2a^2}{\rho^2}\sin\alpha\cos\alpha;$$

b) for an orbit inclined to the equator at an angle $45° < \alpha < 90°$,

$$\Delta\gamma_{1 im} = \frac{e^2 a^2}{2\rho^2}.$$

The expressions for the limit errors in the determination of the radial distance to the center of the planet with consideration of (2) will be:

a) for the polar orbit ($\alpha = 90°$) /290

$$\Delta\rho_{1 im} = A \quad \text{for } B < 0,$$

$$\Delta\rho_{1 im} = A + B \quad \text{for } B > 0;$$

b) for the orbit inclined to the equator at an angle $\alpha \neq 90°$

$$\Delta\rho_{1 im} = A + B\cos^2\alpha \quad \text{for } B < 0,$$

$$\Delta\rho_{1 im} = A + B \quad \text{for } B > 0,$$

where, according to (6),

$$A = \frac{e^2 a^2}{2\rho^2}, \quad B = \frac{e^2}{4\rho}(\rho^2 - 3a^2).$$

Conclusions

1. The investigation described herein of the errors of the horizon sensor due to the effect of the nonsphericity of the planet shows that starting with a radial distance that is $\sqrt{3}$ times greater than the equatorial radius of the planet, the errors in the determination of this distance will increase as the altitude of the flight increases. At lower altitudes the above-stated errors are determined, moreover, by the angle of inclination of the orbit to the equatorial plane.

2. It has been found that the above-stated errors, obtained with a probability of 95%, are close in value to the values of the limit error, which permits in engineering calculations to take into consideration only the latter.

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